Dirichlet-Neumann and Neumann-Neumann Methods

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Introductory Domain Decomposition Short Course
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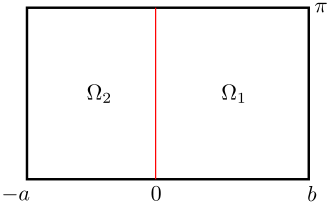
Outline

Outline

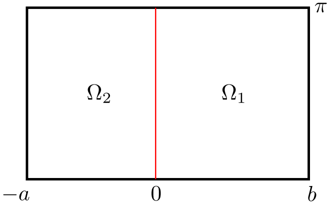
DD methods without overlap



Methods that do not require overlap between subdomains:

1. Optimized Schwarz methods (lecture by Laurence Halpern)
2. Dirichlet-Neumann and Neumann-Neumann methods (this lecture)

DD methods without overlap



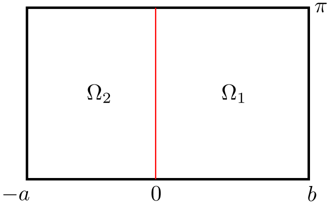
Method converges to monodomain solution if the subdomain solutions have the same function values **and derivatives** across the interface:

$$u_1 = u_2 \qquad \text{on } \Gamma$$

$$\frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2} \qquad \text{on } \Gamma$$

Outline

Dirichlet–Neumann Method (Bjørstad & Widlund 1986)



- Geometry: assume two **non-overlapping** subdomains
- Iterates: solution on the interface $u_\Gamma^1, u_\Gamma^2, \ldots$
- Once interface values converge, solve subdomain problems to get solution everywhere

Dirichlet–Neumann Method (Bjørstad & Widlund 1986)

- Iteration n consists of **two** substeps:
  1. Solve Dirichlet problem on Ω_1, using u_Γ^n as Dirichlet value on Γ

$$\begin{cases} -\Delta u_1^{n+1/2} = f & \text{in } \Omega_1 \\ u_1^{n+1/2} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \\ u_1^{n+1/2} = u_\Gamma^n & \text{on } \Gamma \end{cases}$$

Dirichlet–Neumann Method (Bjørstad & Widlund 1986)

- Iteration n consists of **two** substeps:
  2. Solve Neumann problem on Ω_2 by matching **normal derivatives** with $u_1^{n+1/2}$:

$$\begin{cases} -\Delta u_2^{n+1} = f & \text{in } \Omega_2 \\ u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \\ \dfrac{\partial u_2^{n+1}}{\partial n_2} = -\dfrac{\partial u_1^{n+1/2}}{\partial n_1} & \text{on } \Gamma \end{cases}$$

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- Update interface trace:

$$u_\Gamma^{n+1} = (1-\theta)u_\Gamma^n + \theta u_2^{n+1}|_\Gamma, \qquad 0 < \theta \leq 1$$

Why does this work?

For $i = 1, 2$, define

- the **Dirichlet-to-Neumann map** $DtN_i(\cdot, \cdot)$ so that

$$DtN_i(f, g) = \frac{\partial u_i}{\partial n_i}|_\Gamma,$$

where u_i satisfies $-\Delta u_i = f$ on Ω_i and $u_i = g$ on Γ;

- the **Neumann-to-Dirichlet map** $NtD_i(\cdot, \cdot)$ so that

$$NtD_i(f, g) = u_i|_\Gamma,$$

where u_i satisfies $-\Delta u_i = f$ on Ω_i and $\partial u_i / \partial n_i = g$ on Γ.

Note that for fixed f and i, DtN and NtD are inverses of each other:

$$DtN_i(f, NtD_i(f, g)) = g, \qquad NtD_i(f, DtN_i(f, g)) = g.$$

Why does it work?

Then the Dirichlet-Neumann (DN) method can be written as

$$u_\Gamma^{n+1} = (1-\theta)u_\Gamma^n + \theta NtD_2(f, -DtN_1(f, u_\Gamma^n)).$$

If the method converges, then at the fixed point $u_\Gamma = \lim_{n\to\infty} u_\Gamma^n$, we have

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In other words, we have u_1 and u_2 such that

$$\begin{aligned} -\Delta u_i &= f && \text{on } \Omega_i \\ u_1 = u_2 &= u_\Gamma && \text{on } \Gamma \\ \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} &= 0 && \text{on } \Gamma \end{aligned}$$

$\Longrightarrow$ continuity and smoothness conditions satisfied.

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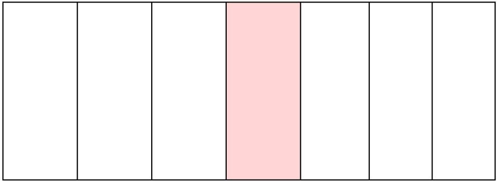
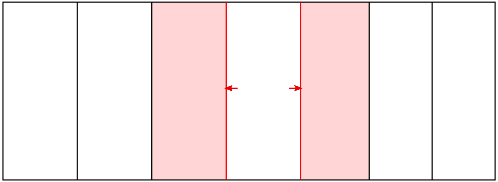
The DN method is a **stationary iteration** for solving $(*)$ for u_Γ.

Multiple subdomains



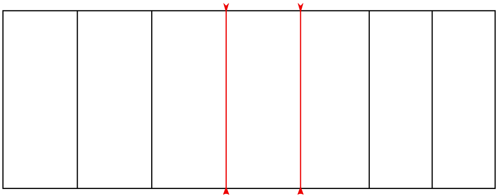
- Start with one subdomain and solve Dirichlet problem

Multiple subdomains



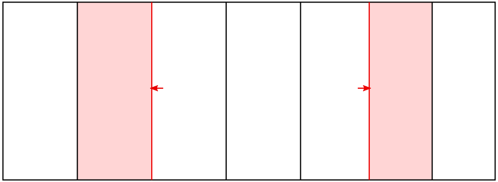
- Start with one subdomain and solve Dirichlet problem
- Solve mixed Dirichlet-Neumann problems on neighbours

Multiple subdomains



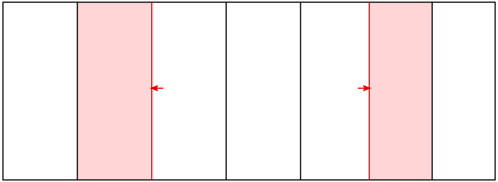
- Start with one subdomain and solve Dirichlet problem
- Solve mixed Dirichlet-Neumann problems on neighbours
- Update interface traces

Multiple subdomains



- Start with one subdomain and solve Dirichlet problem
- Solve mixed Dirichlet-Neumann problems on neighbours
- Update interface traces
- Move on to next set of neighbours, etc.

Multiple subdomains

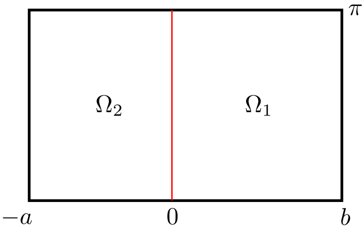


- Start with one subdomain and solve Dirichlet problem
- Solve mixed Dirichlet-Neumann problems on neighbours
- Update interface traces
- Move on to next set of neighbours, etc.

Many configurations possible!

Convergence Analysis by Fourier

- Assume 2D rectangular domain $(-a, b) \times (0, \pi)$
- Two subdomains: $\Omega_1 = (0, b) \times (0, \pi)$, $\Omega_2 = (-a, 0) \times (0, \pi)$



- Solve $-\Delta u = f$ with Dirichlet boundary conditions
- Since problem is linear, assume $f = 0$ and study how the error $e^n := u^n - u$ tends to zero as $n \to \infty$

Convergence Analysis by Fourier

- Expand e_Γ^n into a Fourier sine series:

$$e_\Gamma^n = \sum_{k\geq 1} \hat{e}_\Gamma^n(k)\sin(ky)$$

- Solve $-\Delta e_1^n = 0$ on Ω_1 using separation of variables:

$$e_1^{n+1/2}(x,y) = \sum_{k\geq 1} A_k \sin(ky)\sinh(k(b-x))$$

$$e_1^{n+1/2}(0,y) = \sum_{k\geq 1} A_k \sin(ky)\sinh(kb) = \sum_{k\geq 1} \hat{e}_\Gamma^n(k)\sin(ky)$$

$$\Longrightarrow \quad e_1^{n+1/2}(x,y) = \sum_{k\geq 1} \frac{\sinh(k(b-x))}{\sinh(kb)}\hat{e}_\Gamma^n(k)\sin(ky)$$

Convergence Analysis by Fourier

- Now solve $-\Delta e_2^{n+1} = 0$ on Ω_2 using separation of variables:

$$e_2^{n+1}(x,y) = \sum_{k\geq 1} B_k \sin(ky)\sinh(k(x+a))$$

$$\frac{\partial e_2^{n+1}}{\partial x}(0,y) = \sum_{k\geq 1} kB_k \sin(ky)\cosh(ka)$$

But

$$\frac{\partial e_1^{n+1/2}}{\partial x}(0,y) = -\sum_{k\geq 1} \frac{k\cosh(kb)}{\sinh(kb)}\hat{e}_\Gamma^n(k)\sin(ky)$$

$$\Longrightarrow \quad e_2^{n+1}(x,y) = -\sum_{k\geq 1} \frac{\cosh(kb)\sinh(k(x+a))}{\cosh(ka)\sinh(kb)}\hat{e}_\Gamma^n(k)\sin(ky)$$

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Convergence Analysis by Fourier

- Therefore,

$$
\begin{aligned}
e_\Gamma^{n+1} &= (1-\theta)e_\Gamma^n + \theta e_2^{n+!}(0,y) \\
&= \sum_{k\geq 1}\left[1-\theta\left(1+\frac{\cosh(kb)\sinh(ka)}{\cosh(ka)\sinh(kb)}\right)\right]\hat{e}_\Gamma^n(k)\sin(ky)
\end{aligned}
$$

$$
\Longrightarrow \quad \hat{e}_\Gamma^{n+1}(k) = \underbrace{\left[1-\theta\left(1+\frac{\tanh(ka)}{\tanh(kb)}\right)\right]}_{\rho(k,a,b,\theta)}\hat{e}_\Gamma^n(k)
$$

- Contraction factor depends on
  1. Frequency k,
  2. Geometry, i.e., relative sizes b and a of Ω_1 and Ω_2,
  3. Relaxation parameter θ.
- If $|\rho(k,a,b,\theta)| < 1$ for all k, then $e_\Gamma^n \to 0$ as $n \to \infty$
- If $|\rho(k,a,b,\theta)| > 1$ for some k, then iteration diverges

Symmetric Domains

$$\hat{e}_\Gamma^{n+1}(k) = \rho(k,a,b,\theta)\hat{e}_\Gamma^{n}(k)$$
$$\rho(k,a,b,\theta) = 1 - \theta\left(1 + \frac{\tanh(ka)}{\tanh(kb)}\right)$$

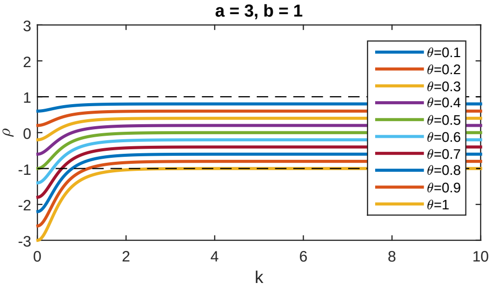
- If Ω_1 and Ω_2 have the same size, i.e., $a = b$, then

$$\rho(k,a,b,\theta) = 1 - 2\theta$$

- Convergence independent of frequency k
- If $\theta = 1/2$, then $e_\Gamma^{n+1} = 0$ for all $n \geq 0$
- Exact solution obtained on Γ after one iteration
- One more subdomain solve to get solution everywhere

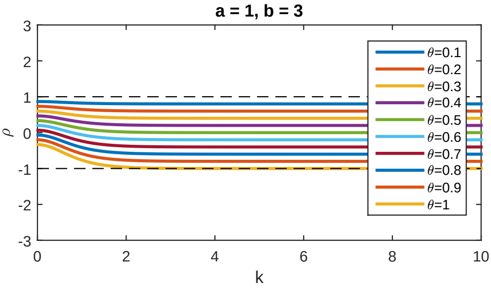
Unsymmetric Domains

Example for $a = 3$, $b = 1$:



Unsymmetric Domains

Example for $a = 1$, $b = 3$:



Unsymmetric Domains

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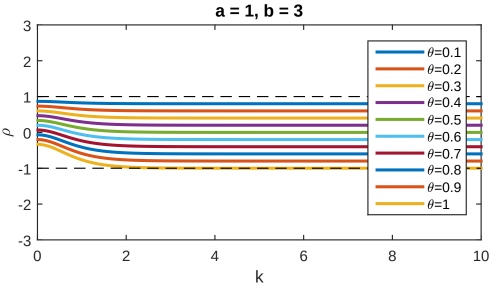


Want to choose θ to minimize $|\rho(k, a, b, \theta)|$ **over all frequencies** k, i.e., solve

$$\min_{\theta} \max_{k \geq 0} |\rho(k, a, b, \theta)|,$$

where

$$\rho(k, a, b, \theta) = 1 - \theta \left(1 + \frac{\tanh(ka)}{\tanh(kb)} \right)$$

Unsymmetric Domains

$$\frac{\partial \rho}{\partial k} = \frac{\theta}{\langle pos.terms \rangle}\left(\frac{\sinh(2ka)}{2a} - \frac{\sinh(2kb)}{2b}\right) = \begin{cases} > 0, & a > b, \\ < 0, & a < b. \end{cases}$$

So $\rho(k, a, b, \theta)$ is monotonic in k with

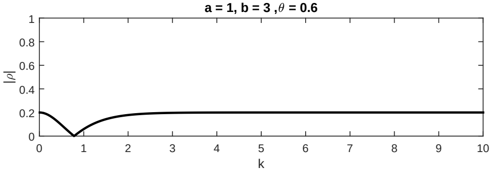
$$\lim_{k \to 0} \rho = (1 - \theta) - \frac{\theta a}{b}, \qquad \lim_{k \to \infty} \rho = 1 - 2\theta.$$

Therefore, the value of θ that minimizes ρ over all k is given by the equioscillation condition

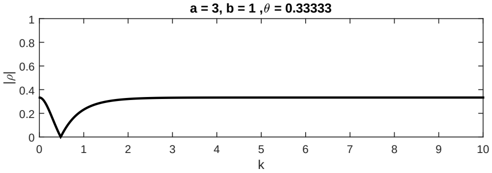
$$\left|\lim_{k \to 0} \rho\right| = \left|\lim_{k \to \infty} \rho\right| \implies \theta_{opt} = \frac{2b}{a + 3b}, \quad \rho_{opt} = \frac{|a - b|}{a + 3b} < 1.$$

Unsymmetric Domains

Example for $a = 1$, $b = 3$: $\theta_{opt} = 0.6$, $\rho_{opt} = 0.2$

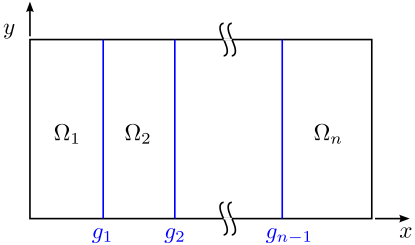


Example for $a = 3$, $b = 1$: $\theta_{opt} = \rho_{opt} = 1/3$



Outline

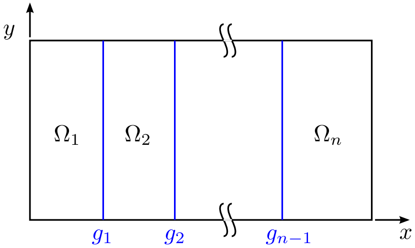
Neumann–Neumann Method (Bourgat, Glowinski, Le Tallec & Vidrascu 1989)



Consider the elliptic problem

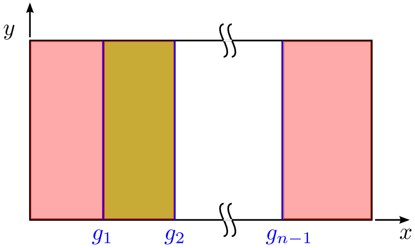
$$-\Delta u = f$$

Neumann–Neumann (NN) Method



Given Dirichlet traces $g_1^n, \ldots, g_{j-1}^n$:

Neumann–Neumann (NN) Method

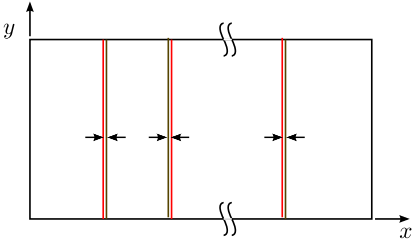


Given Dirichlet traces $g_1^n, \dots, g_{j-1}^n$:

1. Solve Dirichlet problems on $\Omega_1, \dots, \Omega_J$:

$$\begin{aligned} -\Delta u_i^{n+1/2} &= f \qquad \text{on } \Omega_i \\ u_i^{n+1/2} &= g_{i-1} \quad \text{on } \Gamma_{i-1,i} \\ u_i^{n+1/2} &= g_i \qquad \text{on } \Gamma_{i,i+1} \end{aligned}$$

Neumann–Neumann (NN) Method

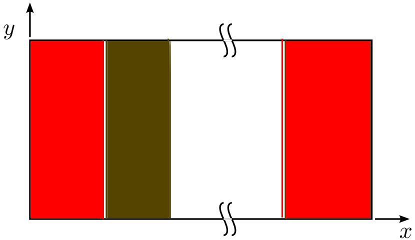


Given Dirichlet traces $g_1^n, \ldots, g_{j-1}^n$:

1. Solve Dirichlet problems on $\Omega_1, \ldots, \Omega_J$:
2. Calculate jumps in Neumann traces along $\Gamma_{i-1,i}$:

$$r_i^{n+1/2} = \frac{\partial u_i^{n+1/2}}{\partial n_i} + \frac{\partial u_{i+1}^{n+1/2}}{\partial n_{i+1}}$$

Neumann–Neumann (NN) Method



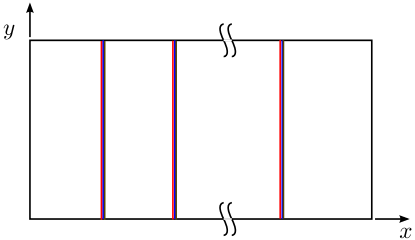
Given Dirichlet traces $g_1^n, \ldots, g_{j-1}^n$:

3. Solve Neumann problems for **corrections** ψ_i^{n+1} on $\Omega_1, \ldots, \Omega_J$:

$$-\Delta \psi_i^{n+1/2} = 0 \qquad \text{on } \Omega_i$$

$$\left.\frac{\partial \psi_i^{n+1/2}}{\partial n_{i-1}}\right|_{\Gamma_{i-1,i}} = r_{i-1}, \qquad \left.\frac{\partial \psi_i^{n+1/2}}{\partial n_i}\right|_{\Gamma_{i,i+1}} = r_i.$$

Neumann–Neumann (NN) Method



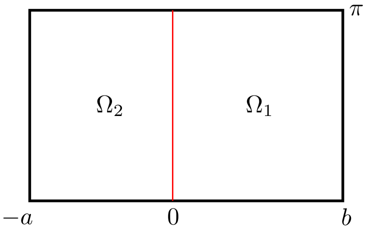
Given Dirichlet traces $g_1^n, \ldots, g_{j-1}^n$:

3. Solve Neumann problems for **corrections** ψ_i^{n+1} on $\Omega_1, \ldots, \Omega_J$:
4. Update Dirichlet traces:

$$g_i^{n+1} = g_i^n - \theta(\psi_i^{n+1}|_{\Gamma_i} + \psi_{i+1}^{n+1}|_{\Gamma_i}).$$

Convergence Analysis

- Assume two rectangular subdomains again:



- Analyze how the error goes to zero for the homogeneous problem

Convergence Analysis

- Fourier analysis gives for the Dirichlet solve

$$e_1^{n+1/2}(x,y) = \sum_{k\geq 1} \frac{\sinh(k(b-x))}{\sinh(kb)} \hat{e}_\Gamma^n(k)\sin(ky)$$

$$e_2^{n+1/2}(x,y) = \sum_{k\geq 1} \frac{\sinh(k(x+a))}{\sinh(ka)} \hat{e}_\Gamma^n(k)\sin(ky)$$

Convergence Analysis

- The normal derivative thus becomes

$$-\frac{\partial e_1}{\partial x}^{n+1/2}(0,y) = \sum_{k\geq 1} \frac{k\cosh(kb)}{\sinh(kb)} \hat{e}_\Gamma^n(k)\sin(ky)$$

$$\frac{\partial e_2}{\partial x}^{n+1/2}(0,y) = \sum_{k\geq 1} \frac{k\cosh(ka)}{\sinh(ka)} \hat{e}_\Gamma^n(k)\sin(ky)$$

$$\Longrightarrow \quad r^{n+1/2} = \sum_{k\geq 1} k\left(\frac{\cosh(ka)}{\sinh(ka)} + \frac{\cosh(kb)}{\sinh(kb)}\right)\hat{e}_\Gamma^n(k)\sin(ky)$$

Convergence Analysis

$$\boxed{r^{n+1/2} = \sum_{k\geq 1} k \left(\frac{\cosh(ka)}{\sinh(ka)} + \frac{\cosh(kb)}{\sinh(kb)} \right) \hat{e}_\Gamma^n(k) \sin(ky)}$$

The corrections ψ_1^{n+1}, ψ_2^{n+1} satisfy

$$\psi_1^{n+1}(x, y) = \sum_{k\geq 1} A_k \sinh(k(b - x)) \sin(ky)$$

$$\psi_2^{n+1}(x, y) = \sum_{k\geq 1} B_k \sinh(k(x + a)) \sin(ky)$$

Convergence Analysis

$$r^{n+1/2} = \sum_{k \geq 1} k \left(\frac{\cosh(ka)}{\sinh(ka)} + \frac{\cosh(kb)}{\sinh(kb)} \right) \hat{e}_\Gamma^n(k) \sin(ky)$$

The corrections ψ_1^{n+1}, ψ_2^{n+1} satisfy

$$-\frac{\partial \psi_1^{n+1}}{\partial x}(0, y) = \sum_{k \geq 1} k A_k \cosh(kb) \sin(ky) = r^{n+1/2}$$

$$\frac{\partial \psi_2^{n+1}}{\partial x}(0, y) = \sum_{k \geq 1} k B_k \cosh(ka) \sin(ky) = r^{n+1/2}$$

Convergence Analysis

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Convergence Analysis

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New interface trace satisfies

$$\begin{aligned} e_\Gamma^{n+1} &= e_\Gamma^n - \theta(\psi_1^{n+1}(0,y) + \psi_2^{n+1}(0,y)) \\ &= \sum_{k\geq 1} \rho(k,a,b,\theta)\hat{e}_\Gamma^n \sin(ky) \end{aligned}$$

where

$$\rho(k,a,b,\theta) = 1 - \theta \left(\frac{\sinh(ka)}{\cosh(ka)} + \frac{\sinh(kb)}{\cosh(kb)} \right) \left(\frac{\cosh(ka)}{\sinh(ka)} + \frac{\cosh(kb)}{\sinh(kb)} \right)$$

Convergence Analysis

$$\boxed{\rho(k,a,b,\theta) = 1 - \theta\left(\frac{\sinh(ka)}{\cosh(ka)} + \frac{\sinh(kb)}{\cosh(kb)}\right)\left(\frac{\cosh(ka)}{\sinh(ka)} + \frac{\cosh(kb)}{\sinh(kb)}\right)}$$

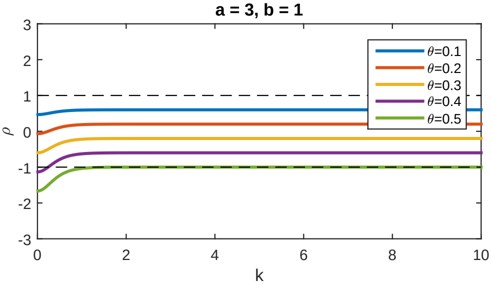
- Expression symmetric in a and b
- If $a = b$, then

$$\rho(k,a,b,\theta) = 1 - 4\theta$$

$\implies$ convergence independent of k
$\implies$ exact convergence after 1 iteration if $\theta = 1/4$

Unsymmetric Domains

Example for $a = 3$, $b = 1$:



- ρ is increasing in k with

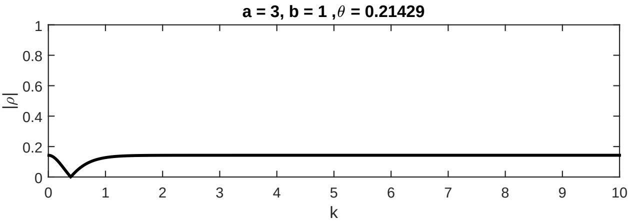
$$\lim_{k\to 0} \rho = (1-2\theta) - \theta\left(\frac{a}{b} + \frac{b}{a}\right), \qquad \lim_{k\to\infty} \rho = 1 - 4\theta.$$

Unsymmetric Domains

- Equioscillation gives

$$\theta_{opt} = \frac{2ab}{(a+b)^2 + 4ab}, \qquad \rho_{opt} = \frac{(a-b)^2}{(a+b)^2 + 4ab}$$

- Example for $a = 3$, $b = 1$: $\theta_{opt} = 3/14$, $\rho_{opt} = 1/7$



Outline

Discrete subdomain problems

Denote the (unknown) Neumann traces by

$$\lambda(v) := \int_\Gamma \frac{\partial u_1}{\partial n_1} v = -\int_\Gamma \frac{\partial u_2}{\partial n_2} v,$$

for all $v \in V_h$. Then integration by parts on Ω_1 gives

$$\int_{\Omega_i} \nabla u_1 \cdot \nabla v - \int_\Gamma \frac{\partial u_1}{\partial n_1} v = \int_{\Omega_i} fv$$

$$A^{(1)}\mathbf{u}_1 = \mathbf{f}_1 + \boldsymbol{\lambda}$$

Similarly, for Ω_2, we have

$$A^{(2)}\mathbf{u}_2 = \mathbf{f}_2 - \boldsymbol{\lambda}.$$

Matrix Formulation

- Partition the problem on Ω_1 into interior and interface unknowns:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{1I}^k \\ \mathbf{u}_{1\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{1I} \\ \mathbf{f}_{1\Gamma} + \lambda^k \end{pmatrix};$$

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- Do the same for Ω_2;

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{2I}^k \\ \mathbf{u}_{2\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{2I} \\ \mathbf{f}_{2\Gamma} - \boldsymbol{\lambda}^k \end{pmatrix};$$

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- This is consistent with the assembled matrix

$$\begin{bmatrix} A_{11}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{1I} \\ \mathbf{u}_{2I} \\ \mathbf{u}_{\Gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{1I} \\ \mathbf{f}_{2I} \\ \mathbf{f}_{1\Gamma} + \mathbf{f}_{2\Gamma} \end{pmatrix}.$$

Dirichlet-Neumann in Matrix Form

Recall subdomain problem on Ω_1:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{1I}^k \\ \mathbf{u}_{1\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{1I} \\ \mathbf{f}_{1\Gamma} + \lambda^k \end{pmatrix}$$

1. Solve Dirichlet problem on Ω_1 with $u_1^k = g^k$ on Γ:

$$A_{II}^{(1)}\mathbf{u}_{1I}^k + A_{I\Gamma}^{(1)}\mathbf{g}^k = \mathbf{f}_{1I}$$

$$\mathbf{u}_{1I}^k = (A_{II}^{(1)})^{-1}(\mathbf{f}_{1I} - A_{I\Gamma}^{(1)}\mathbf{g}^k)$$

Dirichlet-Neumann in Matrix Form

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2. Calculate interface condition λ^k:

$$\lambda^k = A_{\Gamma I}^{(1)} \mathbf{u}_{1I}^k + A_{\Gamma\Gamma}^{(1)} \mathbf{g}^k - \mathbf{f}_{1\Gamma}$$

Dirichlet-Neumann in Matrix Form

Recall subdomain problem on Ω_1:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{1I}^k \\ \mathbf{u}_{1\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{1I} \\ \mathbf{f}_{1\Gamma} + \boldsymbol{\lambda}^k \end{pmatrix}$$

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2. Calculate interface condition $\boldsymbol{\lambda}^k$:

$$\boldsymbol{\lambda}^k = \underbrace{A_{\Gamma I}^{(1)} (A_{II}^{(1)})^{-1} \mathbf{f}_{1I} - f_{1\Gamma}}_{\tilde{\mathbf{f}}_{1\Gamma}} + \underbrace{(A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{\Gamma I}^{(1)} A_{I\Gamma}^{(1)})}_{S_1} \mathbf{g}^k$$

Dirichlet-Neumann in Matrix Form

Recall subdomain problem on Ω_1:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{1I}^k \\ \mathbf{u}_{1\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{1I} \\ \mathbf{f}_{1\Gamma} + \boldsymbol{\lambda}^k \end{pmatrix}$$

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2. Calculate interface condition $\boldsymbol{\lambda}^k$:

$$\boldsymbol{\lambda}^k = \tilde{\mathbf{f}}_{1\Gamma} - S_1 \mathbf{g}^k$$

Note: Dirichlet-to-Neumann map in matrix form!

Dirichlet-Neumann in Matrix Form

3. Solve Neumann problem on Ω_2:

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{2I}^k \\ \mathbf{u}_{2\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{2I} \\ \mathbf{f}_{2\Gamma} - \lambda^k \end{pmatrix}$$

$$A_{II}^{(2)} \mathbf{u}_{2I}^k + A_{I\Gamma}^{(2)} \mathbf{u}_{2\Gamma}^k = \mathbf{f}_{2I}$$

$$A_{\Gamma I}^{(2)} \mathbf{u}_{2I}^k + A_{\Gamma\Gamma}^{(2)} \mathbf{u}_{2\Gamma}^k = \mathbf{f}_{2\Gamma} - \lambda^k$$

Dirichlet-Neumann in Matrix Form

3. Solve Neumann problem on Ω_2:

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$$\mathbf{u}_{2I}^k = (A_{II}^{(2)})^{-1}(\mathbf{f}_{2I} - A_{I\Gamma}^{(2)}\mathbf{u}_{2\Gamma}^k)$$

$$A_{\Gamma I}^{(2)}\mathbf{u}_{2I}^k + A_{\Gamma\Gamma}^{(2)}\mathbf{u}_{2\Gamma}^k = \mathbf{f}_{2\Gamma} - \boldsymbol{\lambda}^k$$

Dirichlet-Neumann in Matrix Form

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$$\underbrace{(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)}(A_{II}^{(2)})^{-1}A_{I\Gamma}^{(2)})}_{S_2} u_{2\Gamma}^k = \underbrace{\mathbf{f}_{2\Gamma} - A_{\Gamma I}^{(2)}(A_{II}^{(2)})^{-1}\mathbf{f}_{2I}}_{\tilde{\mathbf{f}}_2} + \underbrace{\tilde{\mathbf{f}}_1 - S_1\mathbf{g}^k}_{-\lambda^k}$$

Dirichlet-Neumann in Matrix Form

3. Solve Neumann problem on Ω_2:

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4. Update interface trace:

$$\mathbf{g}^{k+1} = (1-\theta)\mathbf{g}^k + \theta\mathbf{u}_{2\Gamma}^k$$

Dirichlet-Neumann in Matrix Form

3. Solve Neumann problem on Ω_2:

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{2I}^k \\ \mathbf{u}_{2\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{2I} \\ \mathbf{f}_{2\Gamma} - \boldsymbol{\lambda}^k \end{pmatrix}$$

$$\mathbf{u}_{2I}^k = (A_{II}^{(2)})^{-1}(\mathbf{f}_{2I} - A_{I\Gamma}^{(2)}\mathbf{u}_{2\Gamma}^k)$$

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4. Update interface trace:

$$\mathbf{g}^{k+1} = [(1-\theta)I - \theta S_2^{-1}S_1]\mathbf{g}^k + \theta S_2^{-1}(\tilde{\mathbf{f}}_1 + \tilde{\mathbf{f}}_2)$$

Dirichlet-Neumann in Matrix Form

3. Solve Neumann problem on Ω_2:

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{2I}^k \\ \mathbf{u}_{2\Gamma}^k \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{2I} \\ \mathbf{f}_{2\Gamma} - \boldsymbol{\lambda}^k \end{pmatrix}$$

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4. Update interface trace:

$$\mathbf{g}^{k+1} = [I - \theta S_2^{-1}(S_1 + S_2)]\mathbf{g}^k + \theta S_2^{-1}(\tilde{\mathbf{f}}_1 + \tilde{\mathbf{f}}_2)$$

Dirichlet-Neumann in Matrix Form

$$\boxed{\mathbf{g}^{k+1} = [I - \theta S_2^{-1}(S_1 + S_2)]\mathbf{g}^k + \theta S_2^{-1}(\tilde{\mathbf{f}}_1 + \tilde{\mathbf{f}}_2)}$$

At convergence, the fixed point **g** satisfies

$$\theta S_2^{-1}(S_1 + S_2)\mathbf{g} = S_2^{-1}(\tilde{f}_1 + \tilde{f}_2)$$

Thus, DN is an iteration for solving the **primal Schur complment problem**

$$(S_1 + S_2)\mathbf{g} = \tilde{f}_1 + \tilde{f}_2$$

with preconditioner $M = \theta^{-1} S_2$.

Neumann-Neumann in Matrix form

By a similar argument, one can show that the Neumann-Neumann method is equivalent to the stationary iteration

$$\boxed{\mathbf{g}^{k+1} = [I - \theta(S_1^{-1} + S_2^{-1})(S_1 + S_2)]\mathbf{g}^k + \theta(S_1^{-1} + S_2^{-1})(\tilde{\mathbf{f}}_1 + \tilde{\mathbf{f}}_2)}$$

So we are still solving the primal Schur system

$$(S_1 + S_2)\mathbf{g} = \tilde{\mathbf{f}}_1 + \tilde{\mathbf{f}}_2,$$

but with the preconditioner $M = \theta(S_1^{-1} + S_2^{-1})$ instead.

Why preconditioning?

- A fixed point iteration can be accelerated using Krylov methods such as **Conjugate Gradient** or **GMRES** (see talk by M.J. Gander)
- For CG, convergence is determined by the ratio of extreme eigenvalues $\lambda_{\max}/\lambda_{\min}$
- For our two-subdomain example with rectangular geometry, it was possible use Fourier to diagonalize A or $M^{-1}A$
- $\lambda_{\min}$ corresponds to low frequency $k \to 0$, and $\lambda_{\max}$ to high frequency $k = k_{\max} = \pi/h \to \infty$

Why preconditioning?

No precond: $$\lambda_k(A) = k\left(\frac{1}{\tanh(ka)} + \frac{1}{\tanh(kb)}\right)$$

$$\lambda_{\min} = \frac{1}{a} + \frac{1}{b}, \qquad \lambda_{\max} \approx 2k_{\max} \to \infty$$

DN: $$\lambda_k(M^{-1}A) = 1 + \frac{\tanh(ka)}{\tanh(kb)}$$

$$\lambda_{\min} = 2, \qquad \lambda_{\max} = 1 + \frac{a}{b} \qquad (a > b)$$

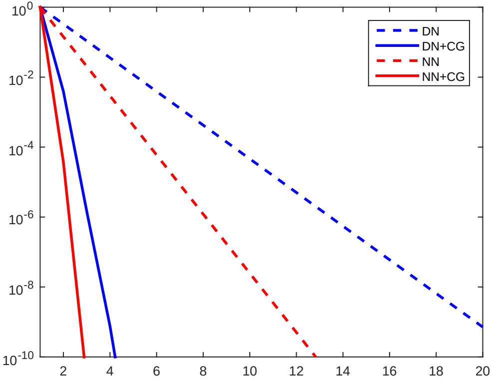
NN: $$\lambda_k(M^{-1}A) = 2 + \frac{\tanh(ka)}{\tanh(kb)} + \frac{\tanh(kb)}{\tanh(ka)}$$

$$\lambda_{min} = 4, \qquad \lambda_{\max} = 2 + \frac{a}{b} + \frac{b}{a}$$

Preconditioned methods converge independently of mesh size!

DN and NN as Preconditioners

Example: $a = 3$, $b = 1$



Dual Schur Problem

Instead of using $\mathbf{u}_\Gamma$ as primary variable, one could use the Neumann trace $\boldsymbol{\lambda}$ instead, where

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{1I} \\ \mathbf{u}_{1\Gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{1I} \\ \mathbf{f}_{1\Gamma} + \boldsymbol{\lambda} \end{pmatrix}$$

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$$S_1 \mathbf{u}_{1\Gamma} = \tilde{\mathbf{f}}_1 + \boldsymbol{\lambda}$$

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$$\mathbf{u}_{1\Gamma} = S_1^{-1}(\tilde{\mathbf{f}}_1 + \boldsymbol{\lambda})$$

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$$\mathbf{u}_{1\Gamma} = S_1^{-1}(\tilde{\mathbf{f}}_1 + \boldsymbol{\lambda})$$

$$\mathbf{u}_{2\Gamma} = S_2^{-1}(\tilde{\mathbf{f}}_2 - \boldsymbol{\lambda})$$

Impose continuity:

$$S_1^{-1}(\tilde{\mathbf{f}}_1 + \boldsymbol{\lambda}) = S_2^{-1}(\tilde{\mathbf{f}}_2 - \lambda)$$

Dual Schur Problem

Instead of using $\mathbf{u}_\Gamma$ as primary variable, one could use the Neumann trace $\boldsymbol{\lambda}$ instead, where

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{1I} \\ \mathbf{u}_{1\Gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{1I} \\ \mathbf{f}_{1\Gamma} + \boldsymbol{\lambda} \end{pmatrix}$$

$$\mathbf{u}_{1\Gamma} = S_1^{-1}(\tilde{\mathbf{f}}_1 + \boldsymbol{\lambda})$$

$$\mathbf{u}_{2\Gamma} = S_2^{-1}(\tilde{\mathbf{f}}_2 - \boldsymbol{\lambda})$$

Impose continuity:

$$(S_1^{-1} + S_2^{-1})\boldsymbol{\lambda} = S_2^{-1}\tilde{\mathbf{f}}_2 - S_1^{-1}\tilde{\mathbf{f}}_1$$

This is the **dual** Schur complement problem, which becomes FETI when preconditioned appropriately.

Summary

1. DN and NN methods: explicitly match derivatives, work for non-overlapping subdomains
2. Interpretation in terms of DtN operators
3. Analysis for two subdomains using Fourier
4. DN and NN as preconditioners $\Longrightarrow$ near grid-independent convergence